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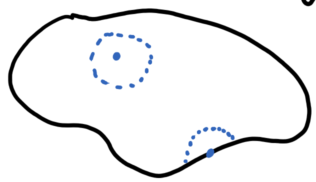
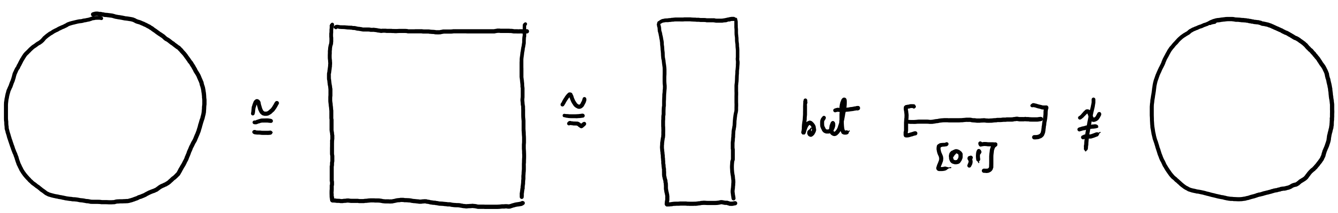
Today - try to understand the group of symmetries of surfaces, that is, the mapping class group
- talk about Dehn twists and how they generate this group
- we will try to understand what the mapping class group is for the disk and torus;

Brief introduction to surfaces

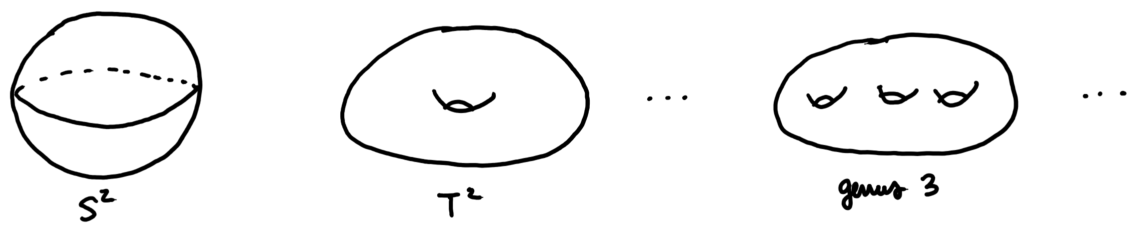
surface - "on the face", intuitively outer layer of an object or boundary between 2 substances such as surface of sea; we think of them as being 2-dim



Definition A homeomorphism between surfaces is a cont f with cont. inverse f^{-1}. (Intuitively stretches and bends).



Examples of surfaces



Theorem (Classification of surfaces) Every compact orientable surface without boundary is homeomorphic to one of the surfaces in the sequence above. If you have boundary then homeomorphism to one of the surfaces above with some interiors of disks removed.

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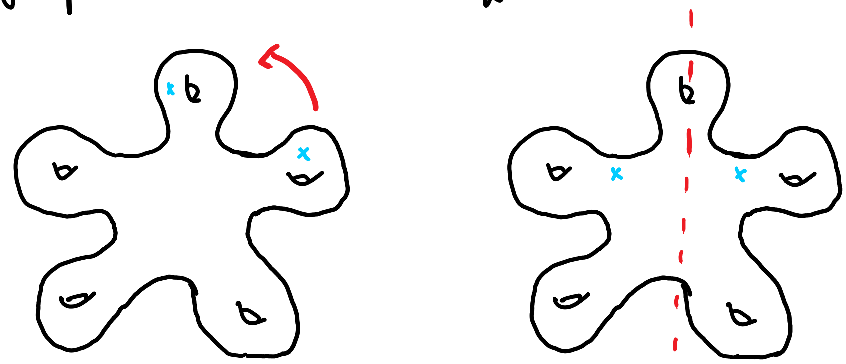
Homeomorphisms of surfaces

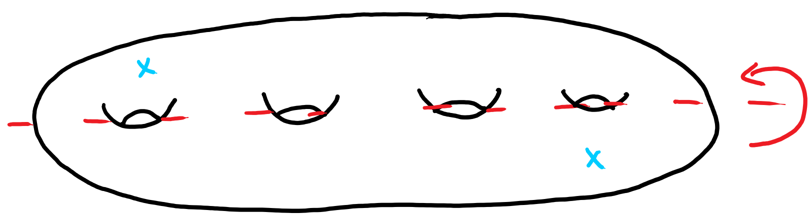
Homeo(S) - self homeo of a surface S - we can think of this group as the symmetries of the surface S.

The mapping class group will be defined as the quotient of a certain subgroup of Homeo(S).

Examples

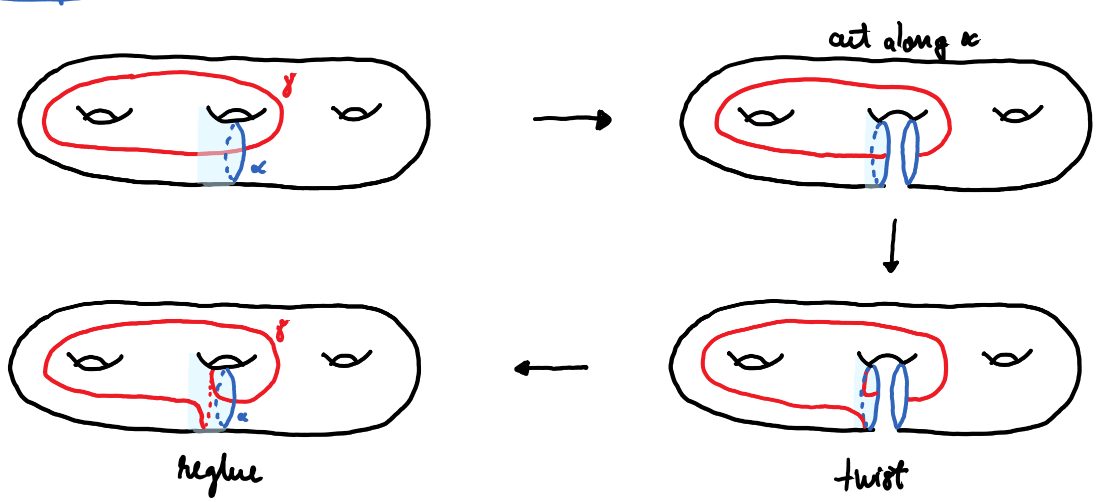


rotation by $\frac{2\pi}{g}$ reflexion



rotation about π

Key example - Dehn twists



Note the simple closed curve γ acquires an extra twist about α!

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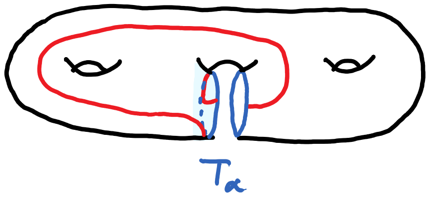
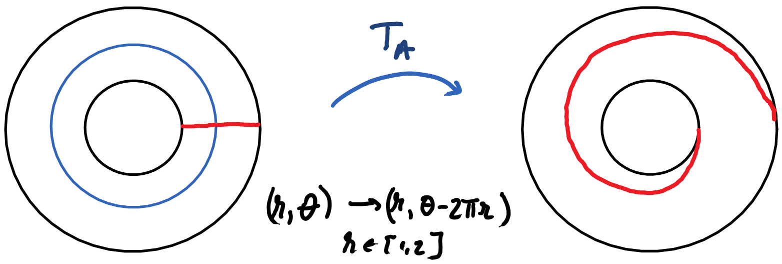
Observe that every simple closed curve in S gives rise to an element of $\mathrm{Homeo}(S)$.
Indeed the twisting and glueing is continuous if you reglue carefully.



We will call this homeo a Dehn twist about α : T_α

We can make this more precise by looking at the annulus A with core α



$(r,\theta) \to (r, \theta - 2\pi r)$
$r \in [1,2]$

Note each point on the boundary is fixed by the map so we can describe the Dehn twist on the surface as extended by identity.

Hence, extension of $T_A \longrightarrow T_\alpha$.

The mapping class group

even $\mathrm{Homeo}(S^2)$ huge!

Trying to understand symmetries of surface $\rightarrow$ Homeo (S) too large!
$\rightarrow$ we would like to declare homeomorphisms that are "similar" to be equiv. while still keeping essential features of Homeo(S).

Idea Look at homotopy classes of such homeomorphisms!

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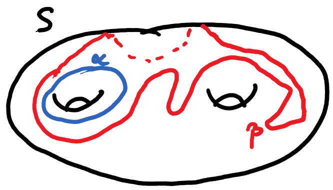
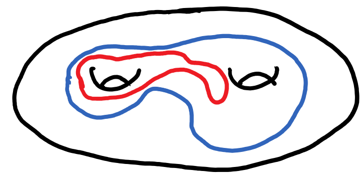
Homotopy - "deforming an object into another"

Examples for curves



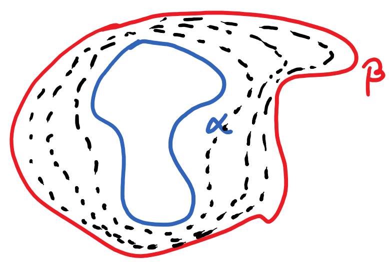
Nonexamples for curves



We can consider α to be the image of some map $S^1 \to S$. Same for β.

Then, a homotopy between them is a continuous map $H: S^1 \times [0,1] \to S$ s.t. $\mathrm{Im}(S^1 \times \{0\})$ is the first curve and $\mathrm{Im}(S^1 \times \{1\})$ is second curve. ↑ "time"

"movie" at $t=0$ we see α and then we watch it slowly get deformed until we see β at $t=1$.



Recall we need to talk about homotopy of homeomorphisms. But how can we deform maps?

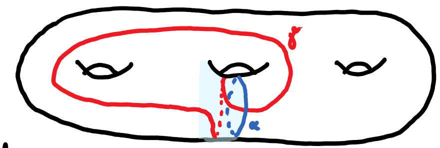
Same idea! Let $f, g \in \mathrm{Homeo}(S)$ they are homotopic if $\exists$ cont $F: S \times [0,1] \to S$ s.t. $F|_{S \times \{0\}} = f$ and $F|_{S \times \{1\}} = g$. Again think of this as a movie.

This is harder to visualize! As an interesting fact this is not a problem for $g \geq 3$. All one needs to check is that $f(\alpha) \simeq h(\alpha)$ $\forall \alpha$ scc.

Definition S compact, orientable surface and let $\mathrm{Homeo}^+(S, \partial S)$ denote the subgroup of $\mathrm{Homeo}(S)$ that preserve orientation and that restrict to id on ∂.

If $h \in \mathrm{Homeo}^+(S, \partial S)$ let $[h]$ all homeo from $S \to S$ that are homotopic to h. The set of all such classes is denoted by $\mathrm{Mod}(S)$ and is called the mapping class group of S.

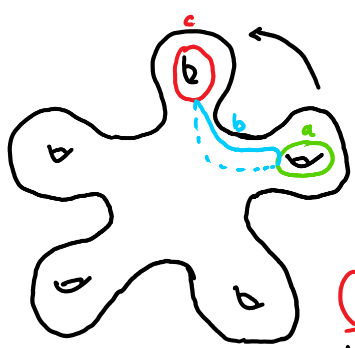
Dehn twists as mapping classes - recall that when defining the Dehn twist corresponding to scc α, the homeo depended on the annulus A and the parametrization of it. Problem! In the context of $\mathrm{Homeo}(S)$ it makes no sense to talk about the Dehn twist about scc α.



However, the homotopy class is independent of such choices. So in $\mathrm{Mod}(S)$ "the" Dehn twist T_α makes sense.

Even better, one can show that if $\alpha \simeq \alpha'$ then $T_\alpha \simeq T_{\alpha'}$. Thus if α is the homotopy class of a scc we can talk about T_a as an element of $\mathrm{Mod}(S)$.

Dehn twists in $\mathrm{Mod}(S)$



Main result $\mathrm{Mod}(S)$ of a compact orientable surface is generated by Dehn twists.

How do we find a product of Dehn twists that takes the homotopy class of a to the one of c? This is not obvious! We will use the following claim to take $a \to b$ and then $b \to c$ via Dehn twists.

Claim If a, b homotopy classes of scc that intersect at one point then $T_a T_b(a) = b$.

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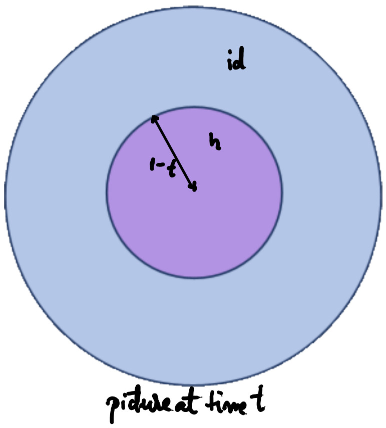
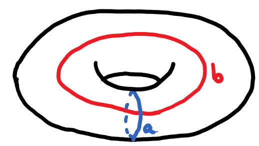
Let us now look at the generators of Mod(S). Recall that for a compact orientable surface Mod(S) is generated by Dehn twists.



D^2 – compact, orientable surface, $g=0$, with 1 boundary comp. Recall that every homeo fix the boundary! But all such maps are homotopic to the identity using the homotopy described in the figure.

Hence, $\mathrm{Mod}(D^2)=0$ so generated by Dehn twists.

T^2 – the torus. Besides being generated by Dehn twists we will see that we have extra str. since $\mathrm{Mod}(T^2) \cong SL(\mathbb{Z},2)$



The idea is that we have a homomorphism $\mathrm{Mod}(T^2) \longrightarrow \mathrm{Aut}(\pi_1(T^2)) \cong GL(2,\mathbb{Z})$ for each class we can pick repr that fixes the base point.

Intuitively, a homeomorphism can be shown to be determined up to homotopy by where the the images of the curves a, b. So if $h(a)$ is (m_1, n_1) and $h(b)$ is (m_2, n_2) we can associate the mapping class of h with $\begin{pmatrix} m_1 & n_1 \\ m_2 & n_2 \end{pmatrix}$.

For $h(a), h(b)$ to intersect once we need $\det = \pm 1$. One can then show that to preserve orientation the determinant has to be 1.
Now T_a corresponds to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and T_b to $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. One can show that these generate $SL(\mathbb{Z},2)$ algebraically.

In general, for S of genus $g \geq 2$ Humphries showed that Mod(S) is generated by the twists about the scc showed in the figure.